On a groupoid construction for actions of certain inverse semigroups

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Abstract

We consider a version of the notion of F-inverse semigroup (studied in the algebraic theory of inverse semigroups). We point out that an action of such an inverse semigroup on a locally compact space has associated a natural groupoid construction, very similar to the one of a transformation group. We discuss examples related to Toeplitz algebras on subsemigroups of discrete groups, to Cuntz-Krieger algebras, and to crossed-products by partial automorphisms in the sense of Exel.

0 Introduction

Let G be a discrete group, let P be a unital subsemigroup of G, and let $\mathcal{W}(G,P)$ denote the Toeplitz (also called Wiener-Hopf) C*-algebra associated to (G,P); i.e., $\mathcal{W}(G,P) \subseteq \mathcal{L}(l^2(P))$ is the C*-algebra generated by the compression to $l^2(P)$ of the left regular representation $\Lambda : l^1(G) \to \mathcal{L}(l^2(G))$.

A powerful tool for studying $\mathcal{W}(G,P)$ is a locally compact groupoid $\mathcal{G}$ introduced by P. Muhly and J. Renault in [7] (in a framework larger than the one of discrete groups), and which was shown in [7] to have $\mathrm{C^*}_{red}(\mathcal{G}) \simeq \mathcal{W}(G,P)$. We shall refer to $\mathcal{G}$ as to the *Wiener-Hopf groupoid* associated to (G,P).

On the other hand let us denote, for every $x \in G$, by $\beta_x : \{t \in P \mid xt \in P\} \to \{s \in P \mid x^{-1}s \in P\}$ the partially defined left translation with x on P; and let us denote by $\mathcal{S}_{G,P}$ the semigroup of bijective transformations between subsets of P (with multiplication given by composition, defined where it makes sense) which is generated by $(\beta_x)_{x\in G}$. Then $\mathcal{S}_{G,P}$ is an *inverse semigroup*, which encodes in some sense the action of G on P by left translations.

The original motivation for this work was to understand the relation between the inverse semigroup $\mathcal{S}_{G,P}$ and the Wiener-Hopf groupoid associated to (G,P). It turns out that:

(A) there exists a remarkable action of $\mathcal{S}_{G,P}$ on a compact space,

and

(B) there exists a very natural procedure of associating a locally compact groupoid to an action of an inverse semigroup in a class which contains $\mathcal{S}_{G,P}$,

such that (A) and (B) together lead to the Wiener-Hopf groupoid.

Moreover, there exists a natural unital $\star$-homomorphism Ψ from $\mathrm{C}^*(\mathcal{S}_{G,P})$ onto $\mathrm{C}^*(\mathcal{G})$, where $\mathrm{C}^*(\mathcal{S}_{G,P})$ denotes the enveloping C*-algebra of $\mathcal{S}_{G,P}$, and $\mathrm{C}^*(\mathcal{G})$ the C*-algebra of the Wiener-Hopf groupoid. Verifying whether Ψ is faithful can be reduced to studying the surjectivity of a map between compact spaces, and actually to comparing two subspaces of $\{0,1\}^G$ (see Corollary 6.4 and Example 6.5 below). This holds when G, viewed as a left-ordered group with positive semigroup P, has some sort of lattice properties (as for instance those considered in [9]), but is not true in general.

Let us now return to the above (A) and (B). By an action of an inverse semigroup $\mathcal{S}$ on a locally compact space Ω we shall understand a unital $\star$-homomorphism from $\mathcal{S}$ into the

inverse semigroup of homeomorphisms between open subsets of Ω.

Related to (A), let us assume for a moment that P does not contain a proper subgroup of G; then the action of $\mathcal{S}_{G,P}$ mentioned at (A), let us call it Φ, is on a space Ω which is a compactification of P. Every $\alpha \in \mathcal{S}_{G,P}$ can thus be viewed as a function between two subsets $Dom(\alpha)$, $Ran(\alpha) \subseteq P \subseteq \Omega$, and we have the remarkable fact that $\Phi(\alpha)$ is the unique continuous extension of α between the closures $\overline{Dom(\alpha)}$ and $\overline{Ran(\alpha)}$ in Ω.

Related to (B): the fact which makes the groupoid construction associated to an action of $\mathcal{S}_{G,P}$ to be very simple, and indeed a straightforward generalization of the groupoid associated to a transformation group, is the following: every element of $\mathcal{S}_{G,P}$, except possibly the zero element, is majorized (in the sense of the usual partial order on an inverse semigroup) by a unique maximal element. Inverse semigroups with this property have been studied in the algebraic theory of inverse semigroups under the name of F-inverse semigroups (see [11], Section VII.6).

There are various examples of F-inverse semigroups coming from the algebraic theory which can be considered (see Example 1.4 below). Given the nature of the present note, it is probably even more interesting to remark that most of the singly generated inverse semigroups have the property under consideration; moreover, the C*-algebra of the groupoid associated to an action of a singly generated inverse semigroup is isomorphic to the corresponding crossed-product C*-algebra by a partial automorphism, in the sense of R. Exel [4].

In [6], A. Kumjian has developed a method which associates a C*-algebra to a *localization*, i.e. to an inverse semigroup $\mathcal{S}$ of homeomorphisms between open subsets of a locally compact space, such that the family of domains $(Dom(\alpha))_{\alpha \in \mathcal{S}}$ is a basis for the topology of the space. The point of view of the present note is slightly different from the one of [6], and the examples we consider do not generally have the named localization property (see Section 5 below). For $\mathcal{S}$ in the common territory of the two approaches, the C*-algebra of the associated groupoid can be shown, however, to be isomorphic to the one constructed in [6] (see Theorem 5.1 below). This is for instance the case for the localizations giving the Cuntz-Krieger C*-algebras $\mathcal{O}_A$.

The paper is subdivided into sections as follows: after recalling in Section 1 some basic definitions, and giving some examples, the groupoid construction is presented in Section 2. In Section 3 we discuss the example related to Toeplitz algebras, and in Section 4 the one related to crossed products by partial automorphisms. Section 5 is devoted to the relation with localizations; in 5.4 the example related to the Cuntz-Krieger algebras is

discussed. Finally, in Section 6 we study the relation between C*($\mathcal{S}$) and the C*-algebra of the associated groupoid (and obtain in particular the facts about $\mathcal{S}_{G,P}$ mentioned at the bottom of page 1).

1 Basic definitions and examples

A semigroup $\mathcal{S}$ is an *inverse semigroup* if for every $\alpha \in \mathcal{S}$ there exists a unique element of $\mathcal{S}$, denoted by α^*, such that $\alpha\alpha^*\alpha = \alpha$, $\alpha^*\alpha\alpha^* = \alpha^*$. For the algebraic facts about inverse semigroups needed in this note, we shall use as a reference the monograph [11]. For T a non-void set, we denote by $\mathcal{I}_T$ the *symmetric inverse semigroup* on T, i.e. the semigroup of all bijections between subsets of T (with multiplication given by composition, defined where it makes sense). This is a generic example, in the sense that every inverse semigroup can be embedded into an $\mathcal{I}_T$. (see [11], IV.1). On an arbitrary inverse semigroup we have a natural partial order, defined by

$$\alpha \leq \beta \overset{def}{\Leftrightarrow} \beta^*\alpha = \alpha^*\alpha, \quad \alpha, \beta \in \mathcal{S}. \tag{1.1}$$

$\alpha \leq \beta$ is equivalent to the fact that α is a restriction of β in every embedding of $\mathcal{S}$ into a symmetric inverse semigroup (see [11], II.1.6 and IV.1.10). An element $\alpha \in \mathcal{S}$ will be called maximal if there is no $\beta \neq \alpha$ in $\mathcal{S}$ such that $\alpha \leq \beta$.

We shall consider only unital inverse semigroups, and the unit will be usually denoted by ϵ. Also, an inverse semigroup may or may not have a zero element, which, if existing, is unique and will be denoted by θ ($\alpha\theta = \theta\alpha = \theta$ for all $\alpha \in \mathcal{S}$). If α is an element of the inverse semigroup $\mathcal{S}$, the expression "α is not zero element for $\mathcal{S}$" will be used to mean that, if $\mathcal{S}$ happens to have a zero element θ, then $\alpha \neq \theta$.

A unital inverse semigroup $\mathcal{S}$ is called an *F-inverse semigroup* if every $\alpha \in \mathcal{S}$ is majorized (in the sense of (1.1)) by a unique maximal element of $\mathcal{S}$ (see [11], Definition VII.5.13). This is essentially the property we are interested in, modulo the following flaw: an F-inverse semigroup with a zero element θ must be a semilattice (Example 1.4.1^o below); this is because the maximal element majorizing θ cannot otherwise be unique. In view of the examples we want to discuss, we slightly weaken the above requirement, as follows:

1.1 Definition A unital inverse semigroup will be called an $\widetilde{F}$*-inverse semigroup* if every $\alpha \in \mathcal{S}$ which is not zero element for $\mathcal{S}$ is majorized by a unique maximal element.

The set of maximal elements of an $\widetilde{F}$-inverse semigroup will be usually denoted by $\mathcal{M}$.

1.2 Example (Toeplitz inverse semigroup) The example which originally motivated these considerations is the following: let G be a group, and let P be a unital subsemigroup of G. For every $x \in G$ put:

$$\beta_x : \{t \in P \mid xt \in P\} \ni t \longrightarrow xt \in \{s \in P \mid x^{-1}s \in P\}. \tag{1.2}$$

Clearly, β_x belongs to the symmetric inverse semigroup $\mathcal{I}_P$. Note that β_x can be the void map θ on P; this happens if and only if $x \in G \setminus PP^{-1}$. Let $\mathcal{S}_{G,P}$ be the subsemigroup of $\mathcal{I}_P$ generated by $\{\beta_x \mid x \in G\}$; this is a unital $\star$ -subsemigroup, since $\beta_x^* = \beta_{x^{-1}}$, $x \in G$, and since β_e ($e =$ unit of G) is the identity map on P. Due to its relation to the Toeplitz algebra associated to G and P, we shall call $\mathcal{S}_{G,P}$ the Toeplitz inverse semigroup of (G, P).

Now, if $PP^{-1} = G$, it turns out that $\mathcal{S}_{G,P}$ has no zero element, and that it is an F-inverse semigroup, with $\mathcal{M} = \{\beta_x \mid x \in PP^{-1}\}$. On the other hand, if $PP^{-1} \neq G$, then $\mathcal{S}_{G,P}$ certainly has a zero element (the void map θ, which is β_x for every $x \in G \setminus PP^{-1}$); in this case, $\mathcal{S}_{G,P}$ still is an $\widetilde{F}$-inverse semigroup, where again $\mathcal{M} = \{\beta_x \mid x \in PP^{-1}\}$ (see Section 3 below).

1.3 Example (singly generated inverse semigroup) Let $\mathcal{S}$ be a unital inverse semigroup which is generated (as unital $\star$-semigroup) by an element $\beta \in \mathcal{S}$. One checks immediately, by embedding $\mathcal{S}$ into a symmetric inverse semigroup, that we have $\beta^m\beta^n \leq \beta^{m+n}$ for every $m, n \in \mathbf{Z}$ (where we make the convention that β^n means $\beta^{\star|n|}$ for $n < 0$). Since any element of $\mathcal{S}$ is a product of β's and β^*'s, it immediately follows that for every $\alpha \in \mathcal{S}$ there exists (at least one) $n \in \mathbf{Z}$ such that $\alpha \leq \beta^n$.

In many examples (of $\mathcal{S}$ generated by β, as above), the $n \in \mathbf{Z}$ such that $\alpha \leq \beta^n$ is uniquely determined, for every $\alpha \in \mathcal{S}$ which is not zero element. This implies that $\mathcal{S}$ is an $\widetilde{F}$-inverse semigroup, with $\mathcal{M} = \{n \in \mathbf{Z} \mid \beta^n$ is not zero element for $\mathcal{S}\}$.

More precisely, let us assume (without loss of generality) that $\mathcal{S} \subseteq \mathcal{I}_T$, the symmetric inverse semigroup on a non-void set T. Denote the subset $\cap_{n\in\mathbf{Z}} Dom(\beta^n)$ of T by T_∞, and put $T_f = T \setminus T_\infty$. Then T_∞ and T_f are invariant for β, and $\beta|T_\infty$ is a bijection. It is easily seen that the only situation when $\mathcal{S}$ can fail to be an $\widetilde{F}$-inverse semigroup is when

T_f, $T_\infty \neq \emptyset$, $\beta|T_\infty$ is periodic and nonconstant, and $\beta|T_f$ is nilpotent, i.e. there is a power of it which is the void map. This does not happen, for instance, in any of the examples discussed in [4, 5].

1.4 Examples Various examples of F-inverse semigroups are studied in the algebraic theory of inverse semigroups. For instance:

1^o Let $\mathcal{E}$ be a unital semilattice, i.e. a unital inverse semigroup all the elements of which are idempotents (see [11], I.3.9). Every $\alpha \in \mathcal{E}$ is selfadjoint (because $\alpha^3 = \alpha$, hence the unique α^* satisfying $\alpha\alpha^*\alpha = \alpha$, $\alpha^*\alpha\alpha^* = \alpha^*$ is $\alpha^* = \alpha$). Hence $\alpha \leq \beta \Leftrightarrow \alpha\beta = \alpha$, $\alpha, \beta \in \mathcal{E}$, which implies that $\mathcal{E}$ is an $\widetilde{F}$-inverse semigroup with $\mathcal{M} = \{\epsilon\}$.

Let us recall here the basic fact that any two idempotents of an arbitrary inverse semigroup $\mathcal{S}$ are commuting (see [11], II.1.2). This implies that, for every inverse semigroup $\mathcal{S}$, the subset of idempotents $\mathcal{S}^{(o)} = \{\alpha \in \mathcal{S} \mid \alpha^2 = \alpha\}$ is a subsemigroup (a semilattice).

2^o Let G be a group, and let $(G_n)_{n\in\mathbf{N}}$ be a sequence of subgroups of G, such that $G_0 = G$ and $G_{n+1} \subseteq G_n$, $n \geq 0$. Consider also $G_\infty = \cap_{n\in\mathbf{N}} G_n$. Define $\mathcal{S} = \cup_{0\leq n\leq\infty} G_n \times \{n\}$, with multiplication

$$(x, m)(y, n) \ = \ (xy, \ \min(m, n)), \quad 0 \leq m, n \leq \infty, \ x \in G_m, y \in G_n.$$

This is an example of a Clifford E-unitary semigroup (see [11], II.2 and III.7.1). It is an F-inverse semigroup, with $\mathcal{M} = (\cup_{n\in\mathbf{N}}(G_n \setminus G_{n+1}) \times \{n\}) \cup (G_\infty \times \{\infty\})$.

3^o Let G be a group and let $\sigma : G \to G$ be an automorphism. Define $\mathcal{S} = \mathbf{N} \times G \times \mathbf{N}$, with multiplication

$$(m, x, n)(p, y, q) = \Big(m - n + \max(n, p), \sigma^{\max(n,p)-n}(x)\sigma^{\max(n,p)-p}(y), q - p + \max(n, p)\Big),$$

for $m, n, p, q \in \mathbf{N}$ and $x, y \in G$. This is an example of a Reilly semigroup (see [11], II.6); it is an F-inverse semigroup, with $\mathcal{M} = \{(m, x, n) \mid m, n \in \mathbf{N}, x \in G, \ \min(m, n) = 0\}$. In the case when G has only one element, $\mathcal{S}(\simeq \mathbf{N} \times \mathbf{N})$ is called the bicyclic semigroup; this was also appearing in the context of Example 1.3 (for $(G, P) = (\mathbf{Z}, \mathbf{N})$, in the notations used there).

4^o It should also be kept in mind that, obviously, every group is an F-inverse semigroup (for an F-inverse semigroup $\mathcal{S}$ we have "$\mathcal{S}$ group $\Leftrightarrow \mathcal{M} = \mathcal{S}$").

1.5 The multiplicative structure on $\mathcal{M}$ Let $\mathcal{S}$ be an $\widetilde{F}$-inverse semigroup, and let $\mathcal{M}$ be the set of maximal elements of $\mathcal{S}$. It is clear that $\mathcal{M}$ contains the unit ϵ of $\mathcal{S}$, and that it is closed under $\star$-operation.

We shall work with a multiplicative structure on $\mathcal{M}$, which is related to the multiplication of $\mathcal{S}$, but can't of course coincide with it. In order to avoid any confusion, we shall use the notation

$$\mathcal{M} \ = \ \{\beta_x \mid x \in M\}, \tag{1.3}$$

where M is some fixed set of the same cardinality with $\mathcal{M}$ (and $M \ni x \to \beta_x \in \mathcal{M}$ is a bijection), and we shall define the multiplicative structure we need as an operation on M. We denote by e the unique element of M such that $\beta_e = \epsilon$ (=unit of $\mathcal{S}$).

The case when $\mathcal{S}$ has not a zero element is quite clear, and well-known (see [11], III.5, VII.6). We define the multiplication on M by

$$x \cdot y \ = \ \text{the unique } z \in M \text{ such that } \beta_x\beta_y \leq \beta_z, \quad x, y \in M. \tag{1.4}$$

Then M becomes a group (e is the unit, and the inverse of $x \in M$ is the unique $u \in M$ such that $\beta_u = \beta_x^*$). The map $\mathcal{S} \to M$ which sends $\alpha \in \mathcal{S}$ into the unique $x \in M$ with $\beta_x \geq \alpha$ is, clearly, a surjective semigroup homomorphism; moreover, it is easily seen that every homomorphism of $\mathcal{S}$ onto a group can be factored by it. Hence M is the maximal group homomorphic image of $\mathcal{S}$.

If $\mathcal{S}$ has a zero element θ, then we have on M only a partially defined multiplication,

$$M^{(2)} = \{(x, y) \in M \times M \mid \beta_x\beta_y \neq \theta\} \ \longrightarrow \ M, \tag{1.5}$$

described by the same rule as in (1.4). Still, for all our purposes this partial multiplication will be as reliable as a group structure (actually, it is not clear whether it wouldn't be always possible to embed it into a group). Remark that:

- We still have a partial associativity property; i.e., if we put

$$M^{(3)} = \{(x, y, z) \in M \times M \times M \mid \beta_x\beta_y\beta_z \neq \theta\}, \tag{1.6}$$

then $(x \cdot y) \cdot z$ and $x \cdot (y \cdot z)$ make sense and are equal for every $(x, y, z) \in M^{(3)}$.

- e still is a unit for M, i.e. $(x, e),\ (e, x) \in M^{(2)}$ and $e \cdot x = x \cdot e = x$ for all $x \in M$.

- For every $x \in M$, the unique $u \in M$ such that $\beta_u = \beta_x^*$, which will be denoted by x^{-1}, has $(x^{-1}, x),\ (x, x^{-1}) \in M^{(2)}$ and $x^{-1} \cdot x = x \cdot x^{-1} = e$. Moreover, it is immediate that $(x^{-1})^{-1} = x$, for every $x \in M$, and that $(x, y) \in M^{(2)} \Rightarrow (y^{-1}, x^{-1}) \in M$, $y^{-1} \cdot x^{-1} = (x \cdot y)^{-1}$.

- If $(x, y) \in M^{(2)}$ and $x \cdot y = z$, then automatically $(x^{-1}, z),\ (z, y^{-1}) \in M^{(2)}$, and $x^{-1} \cdot z = y,\ z \cdot y^{-1} = x$. Indeed, we have $\beta_x\beta_y \neq \theta \Rightarrow \beta_x\beta_y\beta_y^* \neq \theta$ (since $(\beta_x\beta_y\beta_y^*)\beta_y = \beta_x\beta_y$), hence $(x \cdot y) \cdot y^{-1}$ and $x \cdot (y \cdot y^{-1})$ make sense and are equal, by the partial associativity;

but $(x \cdot y) \cdot y^{-1} = z \cdot y^{-1}$, while $x \cdot (y \cdot y^{-1}) = x$. The equality $x^{-1} \cdot z = y$ is proved similarly. Note that, as a consequence, the partial multiplication on M has the cancellation property ($x \cdot y = x \cdot z$ or $y \cdot x = z \cdot x \Rightarrow y = z$).

2 The groupoid construction

2.1 Definition Let $\mathcal{S}$ be a unital inverse semigroup, and let Ω be a locally compact Hausdorff space. By an action of $\mathcal{S}$ on Ω we shall understand a $\star$-homomorphism Φ of $\mathcal{S}$ into the symmetric inverse semigroup of Ω, such that:

(i) for every $\alpha \in \mathcal{S}$, the domain $Dom(\Phi(\alpha))$ and the range $Ran(\Phi(\alpha))$ of $\Phi(\alpha)$ are open in Ω, and $\Phi(\alpha)$ is a homeomorphism between them;

(ii) $\Phi(\epsilon))$ is the identity map on Ω (where ϵ denotes, as usual, the unit of $\mathcal{S}$);

(iii) if $\mathcal{S}$ has a zero element θ, then $\Phi(\theta))$ is the void map on Ω.

For the terminology and basic facts about locally compact groupoids used in this note, we refer the reader to the monograph [12]. The following groupoid construction is a natural generalization to $\widetilde{F}$-inverse semigroups of the very basic example of groupoid associated to a group action ([12], Example I.1.2a).

2.2 Definition Let $\mathcal{S}$ be a unital $\widetilde{F}$-inverse semigroup, and let Φ be an action of $\mathcal{S}$ on the locally compact Hausdorff space Ω. We use the notations related to $\mathcal{S}$ which were introduced in Section 1.5 above (M such that $\mathcal{M} = \{\beta_x \mid x \in M\}$, and the multiplicative structure on M). We then define a groupoid $\mathcal{G}$, as follows:

$$\mathcal{G} \ = \ \{(x, \omega) \mid x \in M, \omega \in Dom(\Phi(\beta_x)) \subseteq \Omega\}. \tag{2.1}$$

The space of units of $\mathcal{G}$ is Ω, and the domain and range of $(x, \omega) \in \mathcal{G}$ are:

$$d(x, \omega) = \omega, \ \ r(x, \omega) = (\Phi(\beta_x))(\omega). \tag{2.2}$$

The multiplication on $\mathcal{G}$ is defined by the rule:

$$(x, \omega)(x', \omega') \ = \ (x \cdot x', \omega'), \tag{2.3}$$

for (x, ω), $(x', \omega') \in \mathcal{G}$ such that $(\Phi(\beta_{x'}))(\omega') = \omega$. (Note: from the latter equality and $\omega \in Dom(\Phi(\beta_x))$ it follows that ω' is in the domain of $\Phi(\beta_x)\Phi(\beta_{x'}) = \Phi(\beta_x \beta_{x'})$; then $\beta_x \beta_{x'}$ can't be zero element for $\mathcal{S}$, and using $\beta_x \beta_{x'} \leq \beta_{x \cdot x'}$ we get $\omega' \in Dom(\Phi(\beta_{x \cdot x'}))$, $(\Phi(\beta_{x \cdot x'}))(\omega') = (\Phi(\beta_x))(\omega)$. Thus the right-hand side of (2.3) is indeed in $\mathcal{G}$, and has $d(x \cdot x', \omega') = d(x', \omega')$, $r(x \cdot x', \omega') = r(x, \omega)$.)

The identity at the unit $\omega \in \Omega$ is (e, ω), with e the unit of M, and the inverse of $(x, \omega) \in \mathcal{G}$ is $(x^{-1}, (\Phi(\beta_x))(\omega))$.

The topology on $\mathcal{G}$ ($\subseteq M \times \Omega$) is the one obtained by restricting to $\mathcal{G}$ the product of the discrete topology on M and the given topology on Ω. This is locally compact (and Hausdorff), since $\mathcal{G}$ is obviously open in $M \times \Omega$. It is immediately seen that multiplication and taking the inverse on $\mathcal{G}$ are continuous, i.e. $\mathcal{G}$ is a locally compact groupoid. The topology induced from $\mathcal{G}$ to the space of identities $\mathcal{G}^{(o)} = \{(e, \omega) \mid \omega \in \Omega\} \simeq \Omega$ is the one of Ω. Moreover, $\mathcal{G}^{(o)}$ is open in $\mathcal{G}$, i.e. $\mathcal{G}$ is an r-discrete groupoid ([12], Definition I.2.6).

2.3 Remark Note that $\mathcal{G}$ in 2.2 is the disjoint union of the sets $\{x\} \times Dom(\Phi(\beta_x))$, $x \in M$, each of them open and closed in $\mathcal{G}$. For every $f \in C_c(\mathcal{G})$, *supp* f is contained in a finite union $\cup_{i=1}^n \{x_i\} \times Dom(\Phi(\beta_{x_i}))$, and we can write $f = \sum_{i=1}^n f h_i$ with h_i the characteristic function of $\{x_i\} \times Dom(\Phi(\beta_{x_i}))$, $1 \leq i \leq n$ ($f h_i \in C_c(\mathcal{G})$, since h_i is continuous). This shows that we have the direct sum decomposition

$$C_c(\mathcal{G}) \; = \; \oplus_{x \in M} \{ f \in C_c(\mathcal{G}) \mid \mathit{supp}\ f \subseteq \{x\} \times Dom(\Phi(\beta_x)) \ \}. \tag{2.4}$$

Note also that from Proposition I.2.8 of [12] it follows that the counting measures form a Haar system on $\mathcal{G}$ (in the sense of [12], Definition I.2.2).

2.4 Remark Let $\mathcal{S}$ be a unital inverse semigroup generated by an element $\beta \in \mathcal{S}$, and such that (as in Example 1.3 above), every $\alpha \in \mathcal{S}$ which is not zero element is majorized by a unique β^n, $n \in \mathbf{Z}$. We have an obvious choice of notation for the set M appearing in equation (1.3) of 1.5. If β^n is not zero element for $\mathcal{S}$, for every $n \in \mathbf{Z}$, then $\mathcal{S}$ has no zero element, $\mathcal{M} = \{\beta^n \mid n \in \mathbf{Z}\}$, and we take $M = \mathbf{Z}$ (the map $M \to \mathcal{M}$ being $n \to \beta^n$). Otherwise, $\mathcal{S}$ must have a zero element θ, and there exists $m \geq 1$ such that $\beta^m \neq \theta = \beta^{m+1}$, $\beta^{\star m} \neq \theta = \beta^{\star (m+1)}$; we then have $\mathcal{M} = \{\beta^n \mid |n| \leq m\}$, and we take $M = \mathbf{Z} \cap [-m, m]$. In any case, the (possibly partially defined) multiplication on M is the usual addition of integers. For Φ an action of $\mathcal{S}$ on a space Ω, as in 2.1, note that the groupoid $\mathcal{G}$ associated

to Φ is

$$\mathcal{G} \;=\; \{(m,\omega) \mid m \in \mathbf{Z}, \omega \in Dom(\Phi(\beta^n))\}, \tag{2.5}$$

even in the case when M is of the form $\mathbf{Z} \cap [-m,m]$ (in this case we have $Dom(\Phi(\beta^n)) = \emptyset$ for $|n| > m$).

Now, remark that $\mathcal{G}$ of (2.5) is an r-discrete locally compact groupoid (with the structure defined in 2.2), even if there is no assumption on β to ensure that $\mathcal{S}$ is an $\widetilde{F}$-inverse semigroup. Indeed, it is seen immediately that the only things required for having a valid groupoid structure on $\mathcal{G}$ of (2.5) are the inequality $\beta^n\beta^m \leq \beta^{n+m}$, $n, m \in \mathbf{Z}$, and the group properties of the addition of the integers, which hold unconditionally.

Moreover, the isomorphism between the C*-algebra of $\mathcal{G}$ in (2.5), on one hand, and the crossed-product C*-algebra (in the sense of [4]) by the partial automorphism given by β, on the other hand, will also turn out to hold with no condition on β (see Theorem 4.1 below).

The above considerations suggest an other simple method of constructing groupoids associated to actions of inverse semigroups. Let G be a group, let $\mathcal{S}$ be a unital inverse semigroup, and let $G \ni x \to \beta_x \in \mathcal{S}$ be a map such that: $\beta_e = \epsilon$, where e is the unit of G and ϵ the one of $\mathcal{S}$; $\beta_{x^{-1}} = \beta_x^*$ for every $x \in G$; $\beta_x\beta_y \leq \beta_{xy}$ for every $x, y \in G$; and for every $\alpha \in \mathcal{S}$ there exists (at least one) $x \in G$ such that $\alpha \leq \beta_x$. (In other words, instead of giving conditions which ensure a multiplicative structure on a remarkable family of elements of $\mathcal{S}$, we impose this from outside.) Then, to an action Φ of $\mathcal{S}$ on a locally compact space Ω, one can associate the locally compact groupoid

$$\mathcal{G} \;=\; \{(x,\omega) \mid x \in G, \omega \in Dom(\Phi(\beta_x))\}, \tag{2.6}$$

with groupoid structure defined as in 2.2. Note that the actions of Toeplitz inverse semigroups can also be considered in this way (in the notations of Example 1.2, we take $G \ni x \to \beta_x \in \mathcal{S}_{G,P}$ to be exactly the one given by equation (1.2)); however, the properties shown in Section 6 below don't hold in general for groupoids of the type (2.6).

3 Example: the Toeplitz inverse semigroup

We now return to the framework of Example 1.3. Consider $G, P, (\beta_x)_{x \in G}$ and $\mathcal{S}_{G,P} \subseteq \mathcal{I}_P$ as in 1.3. We begin by proving the assertions which were made there about $\mathcal{S}_{G,P}$.

3.1 Lemma If $PP^{-1} = G$, then $\mathcal{S}_{G,P}$ has no zero element.

Proof If $\mathcal{S}_{G,P}$ has a zero element θ, then this must be the void map on P (we leave here apart the case when $G = P = \{e\}$). Indeed, θ would otherwise be the identical transformation of a subset of P which is fixed by every β_x, $x \in PP^{-1}$; but for $x \neq e$, β_x has no fixed points.

Hence, since an arbitrary element of $\mathcal{S}_{G,P}$ is a product of β_x's, what we need to show is that for every $n \geq 1$ and $x_1, \ldots, x_n \in G$, the partially defined transformation $\beta_{x_1} \cdots \beta_{x_n}$ on P is non-void.

Let $x_1, \ldots, x_n \in G$ be arbitrary. Since $PP^{-1} = G$, we can find $s_1, \ldots, s_n, t_1, \ldots, t_n \in P$ such that: $x_n = s_n t_n^{-1}$, $x_{n-1}s_n = s_{n-1}t_{n-1}^{-1}$, $x_{n-2}s_{n-1} = s_{n-2}t_{n-2}^{-1}$, $\ldots$, $x_1 s_2 = s_1 t_1^{-1}$. Then $t = t_n \ldots t_1 \in P$ is in the domain of $\beta_{x_1} \cdots \beta_{x_n}$, because, as it is easily checked, $x_j \ldots x_n t = s_j t_{j-1} \ldots t_1 \in P$, $1 \leq j \leq n$. Thus $\beta_{x_1} \cdots \beta_{x_n}$ is indeed non-void. **QED**

3.2 Lemma For every non-void $\alpha \in \mathcal{S}_{G,P}$ there exists a unique $x \in PP^{-1}$ such that $\alpha \leq \beta_x$; this x can be expressed as $\alpha(t)t^{-1}$, with t arbitrary in the domain of α.

Proof Write $\alpha = \beta_{x_1} \cdots \beta_{x_n}$, with $x_1 \cdots x_n \in PP^{-1}$; then $Dom(\alpha) = \{t \in P \mid x_n t, x_{n-1}x_n t, \ldots, x_1 \cdots x_n t \in P\}$, and $\alpha(t) = x_1 \cdots x_n t$ for $t \in Dom(\alpha)$. Putting $x = x_1 \cdots x_n$, we get $\alpha(t)t^{-1} = x$ for every $t \in Dom(\alpha)$ (in particular x is in PP^{-1}). It is clear that $\alpha \leq \beta_x$, and that x is the unique element of PP^{-1} having this property. **QED**

From the above two lemmas it is clear that no matter if $PP^{-1} = G$ or not, $\mathcal{S}_{G,P}$ is an $\widetilde{F}$-inverse semigroup, with $\mathcal{M} = \{\beta_x \mid x \in PP^{-1}\}$. It fits very well the notations of Section 1.5 to take $M = PP^{-1}$. Note also that the multiplication defined on $M = PP^{-1}$ by equation (1.4) of 1.5 coincides with the one coming from the group G; this happens because for any $x, y \in PP^{-1}$ such that $\beta_x\beta_y \neq \theta$, $\beta_x\beta_y$ acts on its domain by left translation with xy.

We now pass to describe a remarkable action of $\mathcal{S}_{G,P}$, which gives via the construction of 2.2 the Wiener-Hopf groupoid $\mathcal{G}$, introduced by P.Muhly and J.Renault in [7]. The space of the action of $\mathcal{S}_{G,P}$ will be

$$\Omega \; = \; clos\{tP^{-1} \mid t \in P\} \subseteq \{0,1\}^G, \tag{3.1}$$

where $\{0,1\}^G$ is identified to the space of all subsets of G. The topology on Ω is the

restriction of the product topology on $\{0,1\}^G$, and is compact and Hausdorff. Note that $P^{-1} \subseteq A \subseteq PP^{-1}$ for every $A \in \Omega$.

3.3 Proposition For every $\alpha \in \mathcal{S}_{G,P}$ we define:

$$\begin{cases} \Phi(\alpha):\ clos\{tP^{-1} \mid t \in Dom(\alpha)\} \to\ clos\{sP^{-1} \mid s \in Ran(\alpha)\}, \\ (\Phi(\alpha))(A) = xA, \end{cases} \tag{3.2}$$

where x in (3.2) is the unique element of PP^{-1} such that $\alpha \leq \beta_x$. (If $\alpha = \theta$, the void map on P, we take by convention $\Phi(\alpha)$ to be the void map on Ω.) Then $\Phi(\alpha)$ makes sense for every $\alpha \in \mathcal{S}_{G,P}$, and Φ is an action of $\mathcal{S}_{G,P}$ on Ω, in the sense of Definition 2.1.

Proof Let $\alpha \in \mathcal{S}_{G,P}$ be non-void, and let x be the unique element of PP^{-1} such that $\alpha \leq \beta_x$. The map $A \to xA$ (=$\{xa \mid x \in A\}$) is continuous on $\{0,1\}^G$, hence $\{A \subseteq G \mid xA \in clos\{sP^{-1} \mid s \in Ran(\alpha)\}\ \}$ is closed. This set contains tP^{-1} for every $t \in Dom(\alpha)$ (because $t \in Dom(\alpha) \Rightarrow xt = \alpha(t) \in Ran(\alpha)$, and on the other hand $x(tP^{-1}) = (xt)P^{-1}$); so $\Phi(\alpha)$ defined in (3.2) makes sense. It is also clear that $\Phi(\alpha^*)$ is an inverse for $\Phi(\alpha)$, which is thus a bijection.

It is useful to note that if $\alpha \in \mathcal{S}_{G,P}$ is written as a product $\beta_{x_1} \cdots \beta_{x_n}$ (for some $n \geq 1$ and $x_1, \dots, x_n \in PP^{-1}$), then we have the equivalent characterization

$$Dom(\Phi(\alpha)) \ = \ \{A \in \Omega \mid A^{-1} \ni x_n,\ x_{n-1}x_n,\ \dots\ , x_1 \cdots x_n\} \tag{3.3}$$

(note: this is valid even if $\alpha = \theta$). Indeed, the right-hand side of (3.3) is immediately seen to be equal to $clos\{tP^{-1} \mid (tP^{-1})^{-1} \ni x_n, x_{n-1}x_n, \dots, x_1 \cdots x_n\} = clos\{tP^{-1} \mid t, x_nt, x_{n-1}x_nt, \dots, x_1 \cdots x_nt \in P\}$, which is exactly $Dom(\Phi(\alpha))$.

As a consequence of (3.3), it is clear that $Dom(\Phi(\alpha))$ (and $Ran(\Phi(\alpha)) = Dom(\Phi(\alpha^*))$, too) is open in Ω for every $\alpha \in \mathcal{S}_{G,P}$.

We are left to show that $\Phi(\alpha)\Phi(\alpha') = \Phi(\alpha\alpha')$ for every $\alpha, \alpha' \in \mathcal{S}_{G,P}$. If one of α, α' is θ, then both $\Phi(\alpha)\Phi(\alpha')$ and $\Phi(\alpha\alpha')$ are the void map on Ω, so we shall assume $\alpha \neq \theta \neq \alpha'$ (but we don't assume $\alpha\alpha' \neq \theta$). We take $x_1, \dots, x_m, y_1, \dots, y_n \in PP^{-1}$ such that $\alpha = \beta_{x_1} \cdots \beta_{x_m}$, $\alpha' = \beta_{y_1} \cdots \beta_{y_n}$; note that the unique $x, y \in PP^{-1}$ such that $\alpha \leq \beta_x$, $\alpha' \leq \beta_y$ must then be $x = x_1 \cdots x_m$ and $y = y_1 \cdots y_n$. Using (3.3) we see that

$$Dom(\Phi(\alpha)\Phi(\alpha')) \ = \ \left\{ A \in \Omega \mid \begin{array}{l} A^{-1} \ni y_n,\ y_{n-1}y_n,\ \dots\ , y_1 \cdots y_n, \\ (yA)^{-1} \ni x_m,\ x_{m-1}x_m,\ \dots\ , x_1 \cdots x_m \end{array} \right\}$$

$$= \left\{ A \in \Omega \mid \begin{array}{l} A^{-1} \ni y_n,\ y_{n-1}y_n,\ \ldots\ ,y_1 \cdots y_n, \\ x_m y_1 \cdots y_n,\ x_{m-1}x_m y_1 \cdots y_n,\ \ldots\ ,x_1 \cdots x_m y_1 \cdots y_n \end{array} \right\}$$

$$= \ Dom(\Phi(\alpha\alpha'))$$

(where the latter equality holds also by (3.3), since $\alpha\alpha' = \beta_{x_1} \cdots \beta_{x_m} \beta_{y_1} \cdots \beta_{y_n}$). If $Dom(\Phi(\alpha)\Phi(\alpha')) = Dom(\Phi(\alpha\alpha'))$ is non-void, it is clear that both transformations act on it by left translation with xy, hence they are equal. **QED**

Note that if $P \cap P^{-1} = \{e\}$, then Ω of (3.1) is a compactification of P (by identifying $t \in P$ with tP^{-1}); in some sense, Ω is obtained from P by adding one point for each "type of convergence to ∞ in P" (compare to the comments in Section 2.3.1 of [8]). Taking into account that a non-void α in $\mathcal{S}_{G,P}$ is actually the left translation with x on $Dom(\alpha)$, with $x \in PP^{-1}$ such that $\alpha \leq \beta_x$, we can interprete the map $\Phi(\alpha)$ of (3.2) as a sort of "compactification of α". [1]

Finally (without having to assume $P \cap P^{-1} = \{e\}$), we have, in view of the characterization (3.3):

$$Dom(\Phi(\beta_x)) \ = \ \{A \in \Omega \mid A \ni x^{-1}\}, \quad x \in PP^{-1}. \tag{3.4}$$

Hence the groupoid associated as in 2.2 to the above action of $\mathcal{S}_{G,P}$ on Ω is:

$$\mathcal{G} \ = \ \{(x, A) \mid A \in \Omega, x \in A^{-1}\}. \tag{3.5}$$

This is exactly the form given to the Wiener-Hopf groupoid in [8], Proposition 2.3.4.

4 Example: singly generated inverse semigroups

Let $\mathcal{S}$ be a unital inverse semigroup, generated (as unital $\star$-semigroup) by an element $\beta \in \mathcal{S}$, and let Φ be an action of $\mathcal{S}$ on the locally compact space Ω (as in 2.1). We denote $\Phi(\beta)$ by $\widetilde{\beta}$. Let $\mathcal{G} = \{(n, \omega) \mid n \in \mathbf{Z},\ \omega \in Dom(\widetilde{\beta}^n)\}$ be the groupoid associated to this action, as in equality (2.5) of Remark 2.4.

[1] One may ask what happens if we don't do any compactification, and just take the obvious action of $\mathcal{S}_{G,P}$ on P. It is immediate that the groupoid associated to this would be the total equivalence relation on P, having thus C*-algebra isomorphic to the compact operators on $l^2(P)$.

On the other hand, if we consider the ideals $I = \{f \in C_o(\Omega) \mid f \equiv 0 \text{ on } \Omega \setminus Ran(\widetilde{\beta})\}$, $J = \{f \in C_o(\Omega) \mid f \equiv 0 \text{ on } \Omega \setminus Dom(\widetilde{\beta})\}$ of $C_o(\Omega)$, then $\widetilde{\beta}$ determines an isomorphism $\theta : I \to J$,

$$(\theta(f))(\omega) \;=\; \begin{cases} f(\widetilde{\beta}(\omega)), & \text{if } \omega \in Dom(\widetilde{\beta}) \\ 0, & \text{otherwise} \end{cases} \quad , \text{ for } f \in I.$$

Thus $\Theta = (\theta, I, J)$ is a partial automorphism of $C_o(\Omega)$, in the sense of R. Exel [4], Definition 3.1, and has attached to it a covariance C*-algebra C*$(C_o(\Omega), \Theta)$ (see [4], Definition 3.7).

4.1 Theorem Assuming Ω second countable, we have that C*(C_o, Θ) and C*$(\mathcal{G})$ are naturally isomorphic.

Proof Following the notations of [4], Section 3, let us put

$$D_n \;=\; \{f \in C_o(\Omega) \mid f \equiv 0 \text{ on } \Omega \setminus Dom(\widetilde{\beta}^n)\}, \quad n \in \mathbf{Z};$$

in other words, D_n is the domain of θ^{-n}. We have in particular $D_o = C_o(\Omega)$, $D_1 = J$, $D_{-1} = I$. Let L be the vector space of sequences $(f_n)_{n\in\mathbf{Z}}$, with $f_n \in D_n$ for every n, and such that $\sum_{n\in\mathbf{Z}} ||f_n|| < \infty$; for $m \in \mathbf{Z}$ and $f \in D_m$ denote by $f\delta_m$ the sequence $(f_n)_{n\in\mathbf{Z}}$ in L which has $f_m = f$ and $f_n = 0$ for $n \neq m$. Then L is given (in [4], p.7) a $\star$-algebra structure which in particular has for $n, m \in \mathbf{Z}$, $f_n \in D_n$, $f_m \in D_m$: $(f_n\delta_n) \star (f_m\delta_m) = g\delta_{n+m}$, $(f_n\delta_n)^* = h\delta_{-n}$, with:

$$g(\omega) \;=\; \begin{cases} f_n(\omega)f_m(\widetilde{\beta}^n(\omega)), & \text{if } \omega \in Dom(\widetilde{\beta}^n) \cap Dom(\widetilde{\beta}^{n+m}) \\ 0, & \text{otherwise,} \end{cases} \tag{4.1}$$

$$h(\omega) \;=\; \begin{cases} \overline{f_n(\widetilde{\beta}^{-n}(\omega))}, & \text{if } \omega \in Dom(\widetilde{\beta}^{-n}) \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

C*$(C_o(\Omega), \Theta)$ is defined as the enveloping C*-algebra of L, with respect to the norm $||(f_n)_{n\in\mathbf{Z}}||_1 = \sum_{n\in\mathbf{Z}} ||f_n||$.

For every $n \in \mathbf{Z}$ let us define

$$\widehat{D_n} \;=\; \{f \in C_c(\Omega) \mid supp\ f \subseteq Dom(\widetilde{\beta}^n)\} \subseteq D_n,$$

and let $\widehat{L}$ be the space of finitely supported sequences $(f_n)_{n\in\mathbf{Z}}$ with the property that $f_n \in \widehat{D_n}$ for every n. Then clearly $\widehat{L}$ is a $\star$-subalgebra of L, dense in $||\cdot||_1$, and C*$(C_o(\Omega), \Theta)$ can also be defined as the enveloping C*-algebra of $(\widehat{L}, ||\cdot||_1)$. Since Ω is assumed to be second countable, the latter normed $\star$-algebra is separable, and we may consider only its representations on separable Hilbert spaces.

Now, every $\star$-representation $\pi : \widehat{L} \to \mathcal{L}(\mathcal{H})$ is automatically contractive with respect to $|| \cdot ||_1$. In order to verify this, it clearly suffices to check that $||\pi(f_n\delta_n)|| \leq ||f_n||$ for every $n \in \mathbf{Z}$ and $f_n \in \widehat{D_n}$. And indeed, one sees immediately (from (4.1),(4.2)) that $(f_n\delta_n) \star (f_n\delta_n)^* = |f_n|^2\delta_o$, hence $||\pi(f_n\delta_n)||^2 = ||\pi(|f_n|^2\delta_o)||$; the latter quantity does not exceed $||\ |f_n|^2\ || = ||f_n||^2$, because π restricted to $\{f\delta_o \mid f \in C_c(\Omega)\} \subseteq \widehat{L}$ gives a $\star$-representation of $C_c(\Omega)$, which is automatically contractive. (The latter assertion holds even if Ω is non-compact, due to the fact that, for every $f \in C_c(\Omega)$, the spectral radius of f in the unitization of $C_c(\Omega)$ equals $||f||$.)

It results that C*$(C_o(\Omega), \Theta)$ is the enveloping C*-algebra of $\widehat{L}$, considered with respect to all the algebraic $\star$-representations of $\widehat{L}$ on separable Hilbert spaces.

On the other hand, due to the separability condition imposed on Ω, it is clear that $\mathcal{G}$ is second countable; hence, by Corollary II.1.22 of [12] (see also Corollaire 4.8 of [13]), C*$(\mathcal{G})$ is the enveloping C*-algebra of $C_c(\mathcal{G})$ with respect to all its (algebraic) $\star$-representations on separable Hilbert spaces. This makes clear that the Theorem will follow as soon as we can prove that the $\star$-algebras $\widehat{L}$ and $C_c(\mathcal{G})$ are isomorphic.

For every $n \in \mathbf{Z}$ and $f_n \in \widehat{D_n}$ let us denote by $\chi_n \otimes f_n$ the restriction to $\mathcal{G} \subseteq \mathbf{Z} \times \Omega$ of the direct product between $\chi_n =$ (characteristic function of $\{n\}$) and f_n. We denote, for every $n \in \mathbf{Z}$, $\mathcal{D}_n = \{\chi_n \otimes f_n \mid f_n \in \widehat{D_n}\} \subseteq C_c(\mathcal{G})$. From the direct sum decomposition of equation (2.4) it is immediate that $C_c(\mathcal{G}) \simeq \oplus_{n\in\mathbf{Z}}\mathcal{D}_n$. (Note that some of the spaces $\mathcal{D}_n$ may be reduced to $\{0\}$, if $\widetilde{\beta}$ is nilpotent; in this case, the corresponding D_n's are also $\{0\}$.) In view of the obvious decomposition $\widehat{L} = \oplus_{n\in\mathbf{Z}}\{f_n\delta_n \mid f_n \in \widehat{D_n}\}$, it becomes clear that $\widehat{L}$ and $C_c(\mathcal{G})$ are naturally isomorphic as vector spaces.

Recalling the definition of the multiplication and $\star$-operation on $C_c(\mathcal{G})$ (from [12], Proposition II.1.1) one gets, for $m, n \in \mathbf{Z}$, $f_n \in \widehat{D_n}$, $f_m \in \widehat{D_m}$, the formulae: $(\chi_n \otimes f_n) \star (\chi_m \otimes f_m)$ $= \chi_{n+m} \otimes g$, $(\chi_n \otimes f_n)^* = \chi_{-n} \otimes h$, where:

$$g(\omega) \ = \begin{cases} f_n(\widetilde{\beta}^m(\omega))f_m(\omega), & \text{if } \omega \in Dom(\widetilde{\beta}^m) \cap Dom(\widetilde{\beta}^{n+m}) \\ 0, & \text{otherwise,} \end{cases} \tag{4.3}$$

and h is the same as in (4.2).

Taking into account the difference of choice which appears in the definition of the multiplication in the two approaches (i.e. in (4.1) vs (4.3)), one has thus to proceed as follows: for every $n \in \mathbf{Z}$ and $f_n \in \widehat{D_n}$ denote by $\Gamma_n(f_n)$ the complex conjugate of the function in (4.2). Then the linear isomorphism $\widehat{L} \to C_c(\mathcal{G})$ determined by $f_n\delta_n \to \chi_{-n} \otimes \Gamma_n(f_n)$, $n \in \mathbf{Z}$, $f_n \in \widehat{D_n}$, is also an isomorphism of $\star$-algebras (the easy verification of this is left to the reader). **QED**

4.2 Remark Crossed-products by partial isomorphisms were used in [5] to approach AF-algebras, and in particular to approach in a very explicit way UHF-algebras. This leads to an interesting point of view on the Glimm groupoid (and more generally on AF-groupoids, defined in [12], Section III.1), as being close, in some sense, to transformation groups.

More precisely, let $(n_i)_{i=0}^{\infty}$ be a sequence of positive integers, and let $X = \prod_{i=0}^{\infty}\{0, 1, \ldots, n_i - 1\}$ have the product topology. Consider, as in [5], the "restricted odometer map" $\beta^* : X \setminus \{(n_1 - 1, n_2 - 1, \ldots)\} \to X \setminus \{(0, 0, \ldots)\}$ which is given by the formal addition with $(1, 0, 0, \ldots)$, with carry over to the right. Let β be the inverse of β^*, let $\mathcal{S}$ be the unital $\star$-semigroup generated by β in $\mathcal{I}_X$, and let $\mathcal{G}$ be the groupoid associated to the action $\Phi(\alpha) \equiv \alpha$ of $\mathcal{S}$ on X. Since β is very close to be a homeomorphism of X, $\mathcal{G}$ is in some sense close to be a transformation group. On the other hand, $\mathcal{G}$ is easily seen to be isomorphic to the Glimm groupoid on $(n_i)_{i=0}^{\infty}$ (defined in [12], p.128); the fact that $C^*(\mathcal{G}) \simeq \mathrm{UHF}((n_i)_{i=0}^{\infty})$ can be verified either this way, or by combining the above Theorem 4.1 with Theorem 3.2 of [5].

5 The relation with localizations (in the sense of Kumjian)

A. Kumjian ([6]) has developed a method which associates a C*-algebra to an inverse semigroup $\mathcal{S}$ of homeomorphisms between open subsets of a locally compact space Ω, such that:

$$(Dom(\alpha))_{\alpha \in \mathcal{S}} \text{ is a basis for the topology of } \Omega. \tag{5.1}$$

Following [6] Definition 2.3, such an inverse semigroup will be called a localization.

The point of view of this note is slightly different from the one of [6]; for instance, the approach to the Glimm groupoid mentioned in Remark 4.2 above is different from the one taken in [6] (see 5.2 below); actually, the inverse semigroup generated by the restricted odometer map is not a localization, and so is also the case for the example described in Section 3 (even in the classical situation when $(G, P) = (\mathbf{Z}, \mathbf{N})$).

Still, the C*-algebra construction of [6] coincides with the C*-algebra of the groupoid defined in 2.2, on the common territory of the two approaches:

5.1 Theorem Let $\mathcal{S}$ be a countable $\widetilde{F}$-inverse semigroup of homeomorphisms between

open subsets of the locally compact space Ω, and assume that $\mathcal{S}$ is a localization. Let $\mathcal{G}$ be the groupoid associated (as in 2.2) to the action $\Phi(\alpha) \equiv \alpha$ of $\mathcal{S}$ on Ω. Then C*($\mathcal{G}$) is isomorphic to the C*-algebra associated to $\mathcal{S}$ in [6].

Proof The C*-algebra associated to $\mathcal{S}$ in [1] [6] is the envelopation of a $\star$-algebra which we will denote (following [6], Section 5) by $C_c(\mathcal{S})$. An argument similar to the one used in the proof of Theorem 4.1 shows that it suffices to prove the isomorphism of $\star$-algebras $C_c(\mathcal{S}) \simeq C_c(\mathcal{G})$.

We now have to go into the details of the definition of $C_c(\mathcal{S})$. Following [6], Notation 5.2, let us put

$$D(\mathcal{S}) \;=\; \oplus_{\alpha\in\mathcal{S}}\{(\alpha, f) \mid f \in C_c(Dom(\alpha))\}, \tag{5.2}$$

where $C_c(Dom(\alpha))$ stands for $\{f \in C_c(\Omega) \mid supp\ f \subseteq Dom(\alpha)\}$, and where the summand in (5.2) corresponding to α is given the linear structure coming from $C_c(Dom(\alpha))$. For $\alpha \in \mathcal{S}$ and $f \in C_c(Dom(\alpha))$ we shall view (α, f) as an element of $D(\mathcal{S})$, in the obvious manner. $D(\mathcal{S})$ is given (in [6], 5.2) a $\star$-algebra structure, such that for $\alpha_1, \alpha_2 \in \mathcal{S}$, $f_1 \in C_c(Dom(\alpha_1))$, $f_2 \in C_c(Dom(\alpha_2))$ we have $(\alpha_1, f_1)(\alpha_2, f_2) = (\alpha_1\alpha_2, g)$, $(\alpha_1, f_1)^* = (\alpha_1^*, h)$, with:

$$g(\omega) \;=\; \begin{cases} f_1(\alpha_2(\omega))f_2(\omega), & \text{if } \omega \in Dom(\alpha_2) \\ 0, & \text{otherwise ,} \end{cases} \tag{5.3}$$

$$h(\omega) \;=\; \begin{cases} \overline{f_1(\alpha_1^*(\omega))}, & \text{if } \omega \in Dom(\alpha_1^*) \\ 0, & \text{otherwise .} \end{cases} \tag{5.4}$$

Then $C_c(\mathcal{S})$ is $D(\mathcal{S})/I_o(\mathcal{S})$ ([6], p.160), where the definition of the ideal $I_o(\mathcal{S}) \subseteq D(\mathcal{S})$ remains to be recalled.

On the other hand, let us also consider the groupoid $\mathcal{G} = \{(x, \omega) \mid\ x \in M, \omega \in Dom(\beta_x)\}$ defined in 2.2, where $\{\beta_x \mid x \in M\}$ is the set of maximal elements of $\mathcal{S}$, as in 1.5. For every $x \in M$ and $f \in C_c(Dom(\beta_x))$ let us denote by $\chi_x \otimes f \in C_c(\mathcal{G})$ the restriction to $\mathcal{G}(\subseteq M \times \Omega)$ of the direct product between $\chi_x =$ (characteristic function of x) and f.

For $\alpha \in \mathcal{S}$ which is not zero element, consider the unique $x \in M$ such that $\alpha \leq \beta_x$; then $Dom(\alpha) \subseteq Dom(\beta_x)$, hence we have a linear map $(\alpha, f) \to \chi_x \otimes f$ from $\{(\alpha, f) \mid f \in C_c(Dom(\alpha))\}$ into $C_c(\mathcal{G})$. The direct sum (after $\alpha \in \mathcal{S}$) of all these linear maps is a linear map $D(\mathcal{S}) \to C_c(\mathcal{G})$, which will be denoted by ρ. Comparing (5.3),(5.4) with the definition of the operations on $C_c(\mathcal{G})$ (given by formulae similar to (4.3),(4.2) of Section 4), one checks

[1] We warn the reader about the unfortunate coincidence that Ω is used in the present paper to denote a space, and in [6] to denote an inverse semigroup.

immediately that ρ is actually a $\star$-algebra homomorphism. ρ is clearly onto, and we are left to show that $Ker\ \rho$ equals $I_o(\mathcal{S})$, the ideal of $D(\mathcal{S})$ mentioned following to (5.4).

The definition of $I_o(\mathcal{S})$ depends on the following notion: a finite family $(\alpha_a, f_a)_{a\in A}$ (with $\alpha_a \in \mathcal{S},\ f_a \in C_c(Dom(\alpha_a))$ for $a \in A$) is called coherent if there are open subsets $(U_a)_{a\in A}$ of Ω such that:

$$\begin{cases} supp\ f_a \subseteq U_a \subseteq Dom(\alpha_a), \quad a \in A, \\ \alpha_a | U_a \cap U_{a'} = \alpha_{a'} | U_a \cap U_{a'} \text{ for } a, a' \in A \text{ having } U_a \cap U_{a'} \neq \emptyset. \end{cases} \tag{5.5}$$

Using this notion, one arrives to $I_o(\mathcal{S})$ in several steps:

- Define $I(\mathcal{S})$ to be the linear span of $\{\sum_{a\in A}(\alpha_a, f_a) \mid (\alpha_a, f_a) \text{ coherent}, \sum_{a\in A} f_a = 0\}$; $I(\mathcal{S})$ is shown to be a two-sided, selfadjoint ideal of $D(\mathcal{S})$ ([6], p. 158).

- For every $\xi \in D(\mathcal{S})$ consider the number

$$|\xi|_o' \ = \ \inf \sum_{b\in B} || \sum_{a\in A_b} f_{a,b} ||_\infty ,$$

where the infimum is taken after all the decompositions $\xi = \sum_{b\in B}\sum_{a\in A_b}(\alpha_{a,b}, f_{a,b})$ (with B and $(A_b)_{b\in B}$ finite sets), such that each of the families $(\alpha_{a,b}, f_{a,b})_{a\in A_b}$ $(b \in B)$ is coherent.

- For every $\xi \in D(\mathcal{S})$ consider the number

$$|\xi|_o \ = \ \inf{}_{\eta\in I(\mathcal{S})} |\xi - \eta|_o' ;$$

then $|\cdot|$ is a $\star$-algebra seminorm on $D(\mathcal{S})$ ([6], p.159).

- Define $I_o(\mathcal{S}) = \{\xi \in D(\mathcal{S})\ ;\ |\xi|_o = 0\}$.

An obvious situation of coherent family $(\alpha_a, f_a)_{a\in A}$ can be obtained by taking all the α_a's $(a \in A)$ to be majorized by the same $\beta_x,\ x \in M$; we shall call such a family majorized-coherent. It is immediate that an element $\xi \in D(\mathcal{S})$ belongs to $Ker\ \rho$ if and only if it is of the form $\sum_{i=1}^n \sum_{a\in A_i}(\alpha_a, f_a)$, where each of the families $(\alpha_a, f_a)_{a\in A_i}$ $(1 \leq i \leq n)$ is majorized-coherent. This makes clear that $Ker\ \rho \subseteq I(\mathcal{S})$; the opposite inclusion is also true, as implied by the following

5.1.1 Lemma If $(\alpha_a, f_a)_{a\in A}$ is a coherent family, with $\sum_{a\in A} f_a = 0$, then there exists a partition $A = A_1 \cup \ldots \cup A_n$ of A such that each of the families $(\alpha_a, f_a)_{a\in A_j}$ $(1 \leq j \leq n)$ is majorized-coherent with $\sum_{a\in A_j} f_a = 0$.

Proof of Lemma 5.1.1 Without loss of generality, we may assume that α_a is non-zero for every $a \in A$; then, for every $a \in A$, we can consider the unique $x(a) \in M$ such

that $\alpha_a \leq \beta_{x(a)}$, and we can partition A such that a, a' are in the same component of the partition if and only if $x(a) = x(a')$.

Pick a family $(U_a)_{a \in A}$ of open subsets of Ω such that (5.5) holds. We will show that $x(a) \neq x(a') \Rightarrow U_a \cap U_{a'} = \emptyset$; this immediately implies that the partition considered in the preceding paragraph satisfies the required conditions.

So, let a, a' be in A such that $U_a \cap U_{a'} \neq \emptyset$. Using (5.1), we can find a non-zero idempotent $\gamma \in \mathcal{S}$ such that $Dom(\gamma) \subseteq U_a \cap U_{a'}$. Then using (5.5) we get $\alpha_a \gamma = \alpha_{a'} \gamma$, non-zero. We have $\alpha_a \gamma \leq \beta_{x(a)}$, $\alpha_{a'} \gamma \leq \beta_{x(a')}$; but the maximal element of $\mathcal{S}$ majorizing $\alpha_a \gamma = \alpha_{a'} \gamma$ is unique, hence $x(a) = x(a')$.

We have thus obtained $I(\mathcal{S}) = Ker\ \rho$. It is obvious that $\xi \in I(\mathcal{S}) \Rightarrow |\xi|_o^{'} = 0 \Rightarrow |\xi|_o = 0$, hence $I_o(\mathcal{S}) \supseteq I(\mathcal{S})$. On the other hand, the inclusion $I_o(\mathcal{S}) \subseteq Ker\ \rho$ will come out from the following

5.1.2 Lemma: If $\xi = \sum_{b \in B} \sum_{a \in A_b} (\alpha_{a,b}, f_{a,b}) \in D(\mathcal{S})$ with $(\alpha_{a,b}, f_{a,b})_{a \in A_b}$ coherent for every $b \in B$, and if $x \in M$, $\omega \in Dom(\beta_x)$, then

$$|(\rho(\xi))(x, \omega)| \ \leq \ \sum_{b \in B} | \sum_{a \in A_b} f_{a,b}(\omega)|. \tag{5.6}$$

Indeed, let us assume the Lemma 5.1.2 true. Fixing ξ and $(\alpha_{a,b}, f_{a,b})_{a,b}$ in (5.6), and letting (x, ω) run in $\mathcal{G}$, gives:

$$||\rho(\xi)||_\infty \ \leq \ \sum_{b \in B} || \sum_{a \in A_b} f_{a,b}||_\infty. \tag{5.7}$$

Then keeping ξ fixed, but taking in (5.7) the infimum after all the possible writings $\xi = \sum_{b \in B} \sum_{a \in A_b} (\alpha_{a,b}, f_{a,b})$, we obtain

$$||\rho(\xi)||_\infty \ \leq |\xi|_o^{'}. \tag{5.8}$$

Finally, replacing in (5.8) ξ by $\xi - \eta$, with $\eta \in I(\mathcal{S})$, using that $\rho(\eta) = 0$, and taking another infimum, leads us to the inequality

$$||\rho(\xi)||_\infty \ \leq \ |\xi|_o; \tag{5.9}$$

this makes the inclusion $I_o(\mathcal{S}) \subseteq Ker\ \rho$ clear.

So we are left to make the

Proof of Lemma 5.1.2 For every $b \in B$, denote $\{a \in A_b \mid \alpha_{a,b} \le \beta_x\}$ by $A_b^{'}$, and $A_b \backslash A_b^{'}$ by $A_b^{''}$. By the definition of ρ we have $(\rho(\xi))(x,\omega) = \sum_{b\in B} \sum_{a\in A_b^{'}} f_{a,b}(\omega)$, which gives $|(\rho(\xi))(x,\omega)| \le \sum_{b\in B} |\sum_{a\in A_b^{'}} f_{a,b}(\omega)|$. It remains to pick an element b of B and verify that $|\sum_{a\in A_b^{'}} f_{a,b}(\omega)| \le |\sum_{a\in A_b} f_{a,b}(\omega)|$; this is clear when $\sum_{a\in A_b^{'}} f_{a,b} = 0$, so we shall assume that there exists at least one $a \in A_b^{'}$ such that $f_{a,b}(\omega) \ne 0$. But, by an argument similar to the one proving Lemma 5.1.1, it is seen that $(\cup_{a\in A_b^{'}} \, supp \; f_{a,b}) \cap (\cup_{a\in A_b^{''}} \, supp \; f_{a,b}) = \emptyset$. Hence if there exists $a \in A_b^{'}$ such that $f_{a,b}(\omega) \ne 0$, then we must have $f_{a,b}(\omega) = 0$ for all $a \in A_b^{''}$, so that $|\sum_{a\in A_b^{'}} f_{a,b}(\omega)| = |\sum_{a\in A_b} f_{a,b}(\omega)|$. **QED**

5.2 Remark In Section 3.6 of [6], a class of localizations with associated AF C*-algebras is constructed. In particular, for a given sequence $(n_i)_{i=0}^{\infty}$ of positive integers, the following localization $\mathcal{L}$ belonging to this class has UHF$((n_i)_{i=0}^{\infty})$ as associated C*-algebra:

- the space on which $\mathcal{L}$ acts is $X = \prod_{i=0}^{\infty}\{0,1,\ldots,n_i-1\}$;
- $\mathcal{L}$ itself is described as

$$\{\gamma(u_o,\ldots,u_k;v_o,\ldots,v_k) \mid k \ge 0,\ 0 \le u_j, v_j \le n_j - 1 \text{ for } 0 \le j \le k\}, \tag{5.10}$$

where for $u_o,\ldots,u_k,v_o,\ldots,v_k$ as above the domain of $\gamma(u_o,\ldots,u_k;v_o,\ldots,v_k)$ is $\{(w_o,w_1,w_2,\ldots) \in X \mid w_j = v_j \text{ for } j \le k\}$, its range is $\{(w_o,w_1,w_2,\ldots) \in X \mid w_j = u_j \text{ for } j \le k\}$, and $\gamma(u_o,\ldots,u_k;v_o,\ldots,v_k)$ acts by replacing the first $k+1$ components of the sequence.

It is immediate that $\mathcal{L}$ of (5.10) is an $\widetilde{F}$-inverse semigroup (its maximal elements are those in (5.10) having $u_k \ne v_k$); hence Theorem 5.1 applies. The groupoid $\mathcal{G}$ associated as in 2.2 to the action of $\mathcal{L}$ on X is easily seen to be (again) the Glimm groupoid. On the other hand, it is obvious that $\mathcal{L}$ is not singly generated, and that (although the acted space X is the same as in Remark 4.2) the intersection between $\mathcal{L}$ and the $\star$-semigroup generated by the restricted odometer is reduced to $\{\epsilon,\theta\}$.

Thus actions of two rather different inverse semigroups can give raise to the same groupoid (and hence the same C*-algebra). In the present case, the reason which makes this happen is that both the inverse semigroup constructions, and the groupoid one, rely on the same method of finding m.a.s.a.'s in AF-algebras ([14], Section 1.1).

5.3 Remark A localization $\mathcal{S}$ on the space Ω is called free ([6], Definition 7.1) if for every $\alpha \in \mathcal{S}$ and $\omega \in Dom(\alpha)$ such that $\alpha(\omega) = \omega$, there exists a neighborhood U of ω in Ω on which α acts as the identity. If in addition $\mathcal{S}$ is assumed to be an $\widetilde{F}$-inverse semigroup,

it follows that every $\alpha \in \mathcal{S}$ which has a fixed point is an idempotent. Indeed, α, ω, U being as above, we may assume (by using (5.1)) that the characteristic function χ of U belongs to $\mathcal{S}$. From $\chi \leq \alpha$ we infer that χ and α are majorized by the same maximal element of $\mathcal{S}$, which can only be ϵ, the unit; but $\alpha \leq \epsilon$ means that α is idempotent.

As an immediate consequence, if $\mathcal{S}$ is a free localization on Ω and also an $\widetilde{F}$-inverse semigroup, then the groupoid $\mathcal{G}$ associated to $\mathcal{S}$ as in 2.2 is principal ([12], Definition I.1.1.1). It is actually clear that $\mathcal{G}$ coincides with the equivalence relation "$\omega \sim \omega' \overset{def}{\Leftrightarrow}$ there is $\alpha \in \mathcal{S}$ such that $\alpha(\omega) = \omega'$" on Ω; hence $\mathcal{G}$ is exactly as in Section 7.2 of [6].

5.4 Example An example of localization which is an $\widetilde{F}$-inverse semigroup, but is not free, is the one described in Section 10.1 of [6], which is related to the Cuntz-Krieger algebras.

Recall from [2] that for an $n \times n$ matrix A with entries $A_{i,j} \in \{0,1\}$, $1 \leq i,j \leq n$, which is irreducible (in the sense that for every i,j there exists $m \in \mathbf{N}$ such that $(A^m)_{i,j} > 0$), and is not a permutation matrix, there exists a unique C*-algebra $\mathcal{O}_A$ generated by n non-zero partial isometries $S_1, \ldots, S_n$ satisfying

$$\begin{cases} (S_iS_i^*)(S_jS_j^*) \ = \ 0, & \text{for } i \neq j \\ S_i^*S_i \ = \ \sum_{j=1}^n A_{i,j}(S_jS_j^*), & \text{for } 1 \leq i \leq n. \end{cases} \tag{5.11}$$

For the matrix A as above, consider the compact space of A-admissible sequences,

$$X_A \ = \ \{(j_o, j_1, j_2, \ldots) \mid 1 \leq j_k \leq n \text{ and } A_{j_k, j_{k+1}} = 1 \text{ for every } k \geq 0\}$$

(with topology obtained by restricting the product topology of $\{0, 1, \ldots, n\}^{\mathbf{N}}$). Then for every $1 \leq m \leq n$ denote by β_m the map:

$$\begin{cases} \{(i_k)_{k\geq 0} \in X_A \mid A_{m,i_o} = 1\} \ \to \ \{(j_k)_{k\geq 0} \in X_A \mid j_o = m\} \\ (i_o, i_1, i_2, \ldots) \ \to \ (m, i_o, i_1, i_2, \ldots); \end{cases}$$

and let $\mathcal{S}_A$ be the unital $\star$-semigroup of homeomorphisms between open compact subsets of X_A which is generated by $\beta_1, \ldots, \beta_n$. $\mathcal{S}_A$ is a localization; indeed, for every finite sequence $j_o, \ldots, j_p$ such that $A_{j_o,j_1} = \ \ldots = A_{j_{p-1},j_p} = 1$, the domain of $\beta_{j_p}^* \cdots \beta_{j_o}^*$ is the set $V_{j_o,\ldots,j_p}$ of sequences in X_A beginning with $j_o, \ldots, j_p$, and the $V_{j_o,\ldots,j_p}$'s are a basis of X_A. Note moreover that the domains of $\beta_1^*, \ldots, \beta_n^*$ form a partition of X_A, on which $\beta_1^*, \ldots, \beta_n^*$ are the restrictions of the one-sided shift on X_A; hence $\mathcal{S}_A$ is as in 10.1 of [6].

On the other hand, it is not difficult to check that $\mathcal{S}_A$ is an $\widetilde{F}$-inverse semigroup; we leave to the reader the verification of the following facts:

5.4.1 Lemma Let $\mathbf{F_n}$ be the free group on generators $g_1, \ldots, g_n$, and let M_A be the set of words $x = g_{i_1} \cdots g_{i_p} g_{j_q}^{-1} \cdots g_{j_1}^{-1} \in \mathbf{F_n}$, with $p, q \geq 0$ and $1 \leq i_1, \ldots, i_p, j_1, \ldots, j_q \leq n$, such that:

(i) $i_p \neq j_q$ (i.e. the word x is in reduced form);

(ii) $A_{i_1,i_2}, \ldots, A_{i_{p-1},i_p}, A_{j_1,j_2}, \ldots, A_{j_{q-1}.j_q}$ are all 1;

(iii) the set $\{1 \leq l \leq n \mid A_{i_p,l} = A_{j_q,l} = 1\}$ is not void.

For every $x = g_{i_1} \cdots g_{i_p} g_{j_q}^{-1} \cdots g_{j_1}^{-1} \in M_A$ denote $\beta_{i_1} \cdots \beta_{i_p} \beta_{j_q}^* \cdots \beta_{j_1}^* \in \mathcal{S}_A$ by β_x. Then the set of maximal elements of $\mathcal{S}_A$ is $\{\beta_x \mid x \in M_A\}$. Moreover, the multiplicative structure on M_A defined as in Section 1.5 above coincides with the one coming from $\mathbf{F_n}$ (i.e. for $x, y \in M_A$ such that $\beta_x\beta_y \neq \theta$, the product in $\mathbf{F_n}$ of x and y is still in M_A, and $\beta_x\beta_y \leq \beta_{xy}$).

From Theorem 5.1 above and from 10.1 of [6] it follows that the groupoid $\mathcal{G}_A$ associated (as in 2.2) to $\mathcal{S}_A$ has $\mathrm{C}^*(\mathcal{G}_A) \simeq \mathcal{O}_A$. Another proof of this isomorphism can be done by using the surjective homomorphism $\Psi : \mathrm{C}^*(\mathcal{S}_A) \to \mathrm{C}^*(\mathcal{G}_A)$ offered by Corollary 6.4 below: since $\beta_1, \ldots, \beta_n$ generate $\mathcal{S}_A$, the partial isometries $\Psi(\beta_1), \ldots, \Psi(\beta_n)$ generate $\mathrm{C}^*(\mathcal{G}_A)$, and it is easily checked that the latter ones satisfy (5.11) (thus $\mathrm{C}^*(\mathcal{G}_A) \simeq \mathcal{O}_A$, due to the uniqueness of $\mathcal{O}_A$).

It should be noted that if the matrix A has all the entries equal to 1, then the groupoid $\mathcal{G}_A$ in the preceding paragraph coincides with the Cuntz groupoid of [12], Section III.2.1.

6 The homomorphism $\mathrm{C}^*(\mathcal{S}) \to \mathrm{C}^*(\mathcal{G})$

Let $\mathcal{S}$ be an $\widetilde{F}$-inverse semigroup, and let $\Phi : \mathcal{S} \to \mathcal{I}_\Omega$ be an action of $\mathcal{S}$ on a space Ω, as in 2.1, having the additional property that $Dom(\Phi(\alpha))$ is compact for every $\alpha \in \mathcal{S}$. (This is true for the examples discussed in Sections 3, 4.2, 5.4.) Note that in particular $\Omega = Dom(\Phi(\epsilon))$ must be compact.

Consider the groupoid $\mathcal{G}$ associated to the action Φ (as in 2.2). The partition $\mathcal{G} = \cup_{x \in M}\{x\} \times Dom(\Phi(\beta_x))$ remarked in 2.2 consists now of open and compact subsets; each of these subsets is a G-set (which means that the restriction to it of the domain map and of the range map are one-to-one - see [12], Definition I.1.10).

Recall now from [12], Definition I.2.10, that the open compact G-sets of $\mathcal{G}$ form (with the pointwise multiplication) an inverse semigroup, called the ample semigroup of $\mathcal{G}$. The partition mentioned in the preceding paragraph (which is indexed by the set $\mathcal{M}$ of maximal

elements of $\mathcal{S}$) "extends" to a homomorphism of inverse semigroups as in the following Lemma, the straightforward proof of which is left to the reader.

6.1 Lemma For every $\alpha \in \mathcal{S}$ define

$$\mathcal{A} = \{x\} \times Dom(\Phi(\alpha)) \subseteq \mathcal{G}, \tag{6.1}$$

where x is the unique element of M such that $\alpha \leq \beta_x$. (If α has a zero element θ, we take by convention $\mathcal{A}(\theta) = \emptyset$, the void set.) Then $\alpha \to \mathcal{A}(\alpha)$ is a unital $\star$-homomorphism from $\mathcal{S}$ into the ample semigroup of $\mathcal{G}$.

Now, it is a basic fact that if A and B are open compact G-sets of an r-discrete groupoid $\mathcal{G}$, and if χ_A, χ_B, χ_{AB} are the characteristic functions of A, B, AB, respectively, then χ_{AB} is the convolution of χ_A and χ_B in $C_c(\mathcal{G})$, and moreover $\chi_A^* = \chi_{A^{-1}}$ in $C_c(\mathcal{G})$. As a consequence, Lemma 6.1 actually gives a unital $\star$-homomorphism $\mathcal{S} \to C_c(\mathcal{G}) \subseteq C^*(\mathcal{G})$. This extends by linearity to the "$\star$-semigroup algebra" $\mathbf{C}[\mathcal{S}]$. (Recall that $\mathbf{C}[\mathcal{S}]$ is the $\star$-algebra having $\mathcal{S}$ (or $\mathcal{S} \setminus \{\theta\}$, in the case with zero element) as a linear basis, and with multiplication and $\star$-operation coming from those of $\mathcal{S}$.) Moreover, the unital $\star$-homomorphism $\mathbf{C}[\mathcal{S}] \to$ C*($\mathcal{G}$) extends by universality to C*($\mathcal{S}$), which is by definition the enveloping C*-algebra of $\mathbf{C}[\mathcal{S}]$.

We pause here to recall that $\mathcal{S}$ has a left regular representation ([1], p. 363); its extension to $\mathbf{C}[\mathcal{S}]$ is faithful ([15]), and this makes C*($\mathcal{S}$) to be a completion of $\mathbf{C}[\mathcal{S}]$, and not of a proper quotient of it. The envelopation of $\mathbf{C}[\mathcal{S}]$ is done after all the unital $\star$-representations of $\mathbf{C}[\mathcal{S}]$ on Hilbert spaces, which correspond to $\star$-representations by partial isometries of $\mathcal{S}$, and are all automatically bounded with respect to the l^1-norm. (See [3] for more details.)

Hence the Lemma 6.1 can be rephrased:

6.2 Proposition There exists a unital $\star$-homomorphism $\Psi : \mathrm{C}^*(\mathcal{S}) \to \mathrm{C}^*(\mathcal{G})$, uniquely determined by

$$\Psi(\alpha) = \chi_{A(\alpha)}, \quad \alpha \in \mathcal{S} \tag{6.2}$$

(with $\mathcal{A}(\alpha)$ as in (6.1), and $\chi_{A(\alpha)} \in C_c(\mathcal{G})$ its characteristic function).

Let us next denote (as in the Example 1.4.1^o above) by $\mathcal{S}^{(o)}$ the subsemigroup of idempotents of $\mathcal{S}$. Then *clos* $sp\{\gamma \mid \gamma \in \mathcal{S}^{(o)}\} \subseteq$ C*($\mathcal{S}$) is, clearly, a unital Abelian C*-subalgebra, which will be denoted by C*($\mathcal{S}^{(o)}$). (It is not difficult to show that, actually, this really

is canonically isomorphic to the C*-algebra of the inverse semigroup $\mathcal{S}^{(o)}$.) On the other hand we denote (following [12], Section II.4) by $C^*(\mathcal{G}^{(o)})$ the unital Abelian C*-subalgebra $\{f \in C_c(\mathcal{G}) \mid \ supp\ f \subseteq \mathcal{G}^{(o)} = \{e\} \times \Omega\}$ of C*($\mathcal{G}$). It is clear that Ψ of (6.2) induces a unital $\star$-homomorphism $\Psi^{(o)} : \mathrm{C}^*(\mathcal{S}^{(o)}) \to \mathrm{C}^*(\mathcal{G}^{(o)})$.

We have the following:

6.3 Theorem In the context of Proposition 6.2, assume in addition that $\mathcal{S}$ is countable and that Ω (the space of the action) is second countable. Then:

1^o Ψ is onto if and only if $\Psi^{(o)}$ is so.

2^o Ψ is an isomorphism if and only if $\Psi^{(0)}$ is so.

Proof If Ψ is onto, then $\Psi^{(o)}$ is onto. Indeed, it is known ([12], Proposition II.4.8, where we compose on the right with the canonical surjection $\mathrm{C}^*(\mathcal{G}) \to \mathrm{C^*}_{red}(\mathcal{G})$) that there exists a conditional expectation $P : \mathrm{C}^*(\mathcal{G}) \to \mathrm{C}^*(\mathcal{G}^{(o)})$, such that $P(f) = \chi_{\mathcal{G}^{(o)}} f$, $f \in C_c(\mathcal{G})$. For $\alpha \in \mathcal{S} \setminus \mathcal{S}^{(o)}$ it is clear that $P(\Psi(\alpha)) = 0$, and this immediately implies $Ran(P \circ \Psi) \subseteq Ran(\Psi^{(o)})$; but $P \circ \Psi$ is onto, since P and Ψ are so, hence $Ran(\Psi^{(o)}) = \mathrm{C}^*(\mathcal{G}^{(o)})$.

It is obvious that Ψ faithful $\Rightarrow$ $\Psi^{(o)}$ faithful, so only the parts "$\Leftarrow$" of 1^o and 2^o above remain to be discussed. We shall use the following

6.3.1 Lemma Let x be in M (the index set for the maximal elements of $\mathcal{S}$) and consider the compact open G-set $\mathcal{A}(\beta_x) = \{x\} \times Dom(\Phi(\beta_x))$ of $\mathcal{G}$. Then Ψ induces a contractive linear map

$$\Psi_x : clos\ sp\{\alpha \in \mathcal{S} \mid \alpha \leq \beta_x\} \ \to \ \{f \in C_c(\mathcal{G}) \mid \ supp\ f \subseteq \mathcal{A}(\beta_x)\}, \tag{6.3}$$

where the latter set is a closed linear subspace of C*($\mathcal{G}$). If $\Psi^{(o)}$ is onto, then Ψ_x is onto; if $\Psi^{(o)}$ is an isomorphism, then Ψ_x is an isomorphism.

Proof of Lemma 6.3.1 If $\mathcal{C}$ is a C*-algebra , $\mathcal{C}_o \subseteq \mathcal{C}$ is a C*-subalgebra, and $w \in \mathcal{C}$ is a partial isometry such that $w^*w \in \mathcal{C}_o$, then $w\mathcal{C}_o$ is a closed linear subspace of $\mathcal{C}$ (for instance because it can be written as $\{c \in \mathcal{C} \mid ww^*c = c, w^*c \in \mathcal{C}_o\}$). Assume in addition that we also have $\mathcal{B}_o \subseteq \mathcal{B}$ C*-algebras, $v \in \mathcal{B}$ partial isometry with $v^*v \in \mathcal{B}_o$, and $\Psi : \mathcal{B} \to \mathcal{C}$ $\star$-homomorphism such that $\Psi(\mathcal{B}_o) \subseteq \mathcal{C}_o$, $\Psi(v) = w$. Then: (a) $\Psi(v\mathcal{B}_o) \subseteq w\mathcal{C}_o$; (b) $\Psi(\mathcal{B}_o) = \mathcal{C}_o$ $\Rightarrow \Psi(v\mathcal{B}_o) = w\mathcal{C}_o$; (c) $\Psi|\mathcal{B}_o$ faithful $\Rightarrow \Psi|v\mathcal{B}_o$ faithful. Indeed, (a),(b) are clear, while (c)

comes out from the norm evaluation:

$$||\Psi(vb)||^2 = ||\Psi b^*(v^*v)b|| \overset{(\star)}{=} ||b^*(v^*v)b|| = ||vb||^2, \quad b \in \mathcal{B}_o,$$

(where at $(\star)$ we used the faithfulness of $\Psi|\mathcal{B}^o$).

In our context, we have to put $\mathcal{B} = \mathrm{C}^*(\mathcal{S})$, $\mathcal{B}_o = \mathrm{C}^*(\mathcal{S}^{(o)})$, $v = \beta_x$ and $\mathcal{C} = \mathrm{C}^*(\mathcal{G})$, $\mathcal{C}_o = \mathrm{C}^*(\mathcal{G}^{(o)})$, $w = \chi_{\mathcal{A}(\beta_x)}$ (Ψ is Ψ).

We return to the proof of the Theorem. If $\Psi^{(o)}$ is onto, then from Lemma 6.3.1 and the direct sum decomposition of (2.4) it is clear that $Ran\Psi \supseteq C_c(\mathcal{G})$, which is dense in $\mathrm{C}^*(\mathcal{G})$; hence Ψ is onto.

If $\Psi^{(o)}$ is an isomorphism, then (by the Lemma) every Ψ_x of (6.3) is a linear isomorphism. The family of linear maps $(\Psi_x^{-1})_{x\in M}$ defines (due to the direct sum decomposition (2.4)) a linear map $\rho : C_c(\mathcal{G}) \to \mathrm{C}^*(\mathcal{S})$. This is easily checked to be a unital $\star$-homomorphism of $C_c(\mathcal{G})$ (one uses the corresponding properties of Ψ). Now, due to the separability hypothesis made in the Theorem, the groupoid $\mathcal{G}$ is second countable, and we can (exactly as we did in the proofs of the Theorems 4.1 and 5.1) extend ρ to a unital C*-algebra homomorphism $\widetilde{\rho} : \mathrm{C}^*(\mathcal{G}) \to \mathrm{C}^*(\mathcal{S})$. $\widetilde{\rho}$ clearly is an inverse for Ψ, which is hence an isomorphism. **QED**

The homomorphism $\Psi^{(o)}$ of Theorem 6.3 has a natural interpretation as a map between compact spaces. Indeed, let Z denote the spectrum of the (unital and Abelian) C*-algebra $\mathrm{C}^*(\mathcal{S}^{(o)})$; this is clearly homomorphic (and will be henceforth identified) to the space of multiplicative functions $\zeta : \mathcal{S}^{(o)} \to \{0,1\}$, such that $\zeta(\epsilon) = 1$, $\zeta(\theta) = 0$ (where ϵ is the unit of $\mathcal{S}$, and θ its zero element - if it exists). On the other hand, the convolution of the functions in $\mathrm{C}^*(\mathcal{G}^{(o)})$ coincides with their pointwise product, which makes clear that the spectrum of $\mathrm{C}^*(\mathcal{G}^{(o)})$ is canonically identified to Ω.

Every $\omega \in \Omega$ gives a character $\zeta_\omega \in Z$, determined by $\zeta_\omega(\gamma) = 1$, if $\omega \in Dom(\Phi(\gamma))$, and $\zeta_\omega(\gamma) = 0$ otherwise ($\gamma \in \mathcal{S}^{(o)}$). The map

$$\psi^{(o)} : \Omega \to Z, \quad \psi^{(o)}(\omega) = \zeta_\omega \tag{6.4}$$

is continuous, because $Dom(\Phi(\gamma))$ is open and compact for every $\gamma \in \mathcal{S}^{(o)}$. It is an immediate verification (left to the reader) that $\psi^{(o)}$ is the map between character spaces corresponding to $\Psi^{(o)} : \mathrm{C}^*(\mathcal{S}^{(o)}) \to \mathrm{C}^*(\mathcal{G}^{(o)})$ of Theorem 6.3. Hence we get:

6.4 Corollary In the context of Theorem 6.3, the homomorphism $\Psi : C^*(\mathcal{S}) \to C^*(\mathcal{G})$ is onto if and only if $\psi^{(o)} : \Omega \to Z$ of (6.4) is one-to one, i.e. if and only if the subsets

$(Dom(\Phi(\alpha)))_{\alpha\in\mathcal{S}}$ separate the points of Ω. Moreover, Ψ is an isomorphism if and only if $\psi^{(o)}$ is bijective.

6.5 Example Let us see how the Corollary 6.4 applies in the context of Toeplitz inverse semigroups. Consider G, P, $(\beta_x)_{x\in G}$, $\mathcal{S}_{G,P}$ as in Example 1.3, and let $\Phi : \mathcal{S}_{G,P} \to \mathcal{I}_\Omega$ be the action defined in Proposition 3.3. From equation (3.3) it is clear that $Dom(\Phi(\alpha))$ is open and compact in Ω, for every $\alpha \in \mathcal{S}_{G,P}$, hence the considerations made in this section can be applied. Moreover, the family of subsets $Dom(\Phi(\alpha))_{\alpha\in\mathcal{S}_{G,P}}$ separates the points of Ω. Indeed, for $A_1 \neq A_2$ in Ω we can take an $x \in (A_1 \setminus A_2) \cup (A_2 \setminus A_1) \subseteq PP^{-1}$ and it is obvious from equation (3.4) in Section 3 that $Dom(\Phi(\beta_x))$ will separate A_1 from A_2.

Hence, we can construct the natural homomorphism $\Psi : \mathrm{C}^*(\mathcal{S}_{G,P}) \to \mathrm{C}^*(\mathcal{G})$, determined by the equation (6.2), and moreover, Ψ is always surjective.

From Corollary 6.4 it also comes out that Ψ will be an isomorphism if and only if the natural map $\psi^{(0)}$ from Ω to the space Z of characters of $\mathrm{C}^*(\mathcal{S}^{(o)}_{G,P})$ is onto. Concerning this, we mention without proof the following facts:

(a) Z can be naturally identified to a subspace of $\{0,1\}^G$, in such a way that $\psi^{(o)} : \Omega \to Z$ becomes an inclusion. This comes from the fact that $\mathcal{S}^{(o)}_{G,P}$ can be shown to be generated by the family $(\beta_x^*\beta_x)_{x\in PP^{-1}}$; hence a multiplicative function on $\mathcal{S}^{(o)}_{G,P}$ is determined by its values on this family, and we get an embedding $\tau : Z \to \{0,1\}^G$, defined by:

$$\tau(\zeta) = \{x \in PP^{-1} \mid \zeta(\beta_x\beta_x^*) = 1\}. \tag{6.5}$$

It turns out that $\Omega \overset{\psi^{(o)}}{\to} Z \overset{\tau}{\to} \{0,1\}^G$ is the inclusion of Ω into $\{0,1\}^G$.

(b) It is well-known that if $\mathcal{E}$ is a unital semilattice, then for every $\gamma \in \mathcal{E}$ which is not zero element we have a character ζ_γ of $\mathrm{C}^*(\mathcal{E})$, determined by: $\zeta_\gamma(\gamma') = 1$, if $\gamma' \geq \gamma$, $\zeta_\gamma(\gamma') = 0$, if $\gamma' \ngeq \gamma$ $(\gamma' \in \mathcal{E})$; moreover, the family of the characters $(\zeta_\gamma)_\gamma$ is dense in the spectrum of $\mathrm{C}^*(\mathcal{E})$. This offers the possibility of writing explicitly a dense subset of $\tau(Z)$, i.e. of Z identified inside $\{0,1\}^G$ as in the preceding paragraph. The dense subset is $\{B_{x_1,\ldots,x_n} \mid n \geq 1, x_1,\ldots,x_n \in PP^{-1}\}$, where

$$\begin{aligned} B_{x_1,\ldots,x_n} &= \{x \in PP^{-1} \mid \beta_x\beta_x^* \geq (\beta_{x_1}^*\beta_{x_1})\cdots(\beta_{x_n}^*\beta_{x_n})\} \\ &= \{x \in PP^{-1} \mid xP \supseteq P \cap x_1^{-1}P \cap \ldots \cap x_n^{-1}P\}. \end{aligned} \tag{6.6}$$

(c) Let us denote by "$\prec$" the left-invariant partial pre-order determined by P on G, i.e. $x \prec y \overset{def}{\Leftrightarrow} x^{-1}y \in P$, for $x, y \in G$. We shall call (G,P) "quasi-lattice ordered" if $P \cap P^{-1} = \{e\}$ and if:

- every $x \in G$ having upper bounds in P (equivalently, $x \in PP^{-1}$) has a least upper bound in P, denoted by $\sigma(x)$;

- every $s, t \in P$ having common upper bounds also have a least common upper bound, denoted by $\sigma(s,t)$.

The class of partially left-ordered groups satisfying these conditions contains the totally left-ordered groups, and is closed under direct products, semidirect products by order-preserving automorphisms, and free products (see [9]), Example 2.3).

For (G,P) quasi-lattice ordered and such that, say, $PP^{-1} \neq G$, the Toeplitz inverse semigroup $\mathcal{S}_{G,P}$ is isomorphic to $(P \times P) \cup \{\theta\}$, with multiplication and $*$-operation:

$$\left\{ \begin{array}{l} (s,t)(u,v) = \left\{ \begin{array}{ll} (st^{-1}\sigma(t,u), vu^{-1}\sigma(t,u)), & \text{if } t,u \text{ have common upper bounds} \\ \theta, & \text{otherwise} \end{array} \right. \\ \theta(s,t) = (s,t)\theta = \theta = \theta^* = \theta^2 \\ (s,t)^* = (t,s) \end{array} \right. \tag{6.7}$$

Indeed, $(s,t) \to \beta_s \beta_t^*$, $\theta \to \theta$, is easily seen to define an isomorphism between $(P \times P) \cup \{\theta\}$ and $\mathcal{S}_{G,P}$. (In the case when $PP^{-1} = G$, we have a similar isomorphism $P \times P \to \mathcal{S}_{G,P}$.)

If (G,P) is quasi-lattice ordered then it is immediately verified, using the particular form of $\mathcal{S}_{G,P}$, that the sets in (6.6) belong indeed to Ω. Hence in this case we have a canonical isomorphism $\mathrm{C}^*(\mathcal{S}_{G,P}) \simeq \mathrm{C}^*(\mathcal{G})$.

(d) In general, the canonical homomorphism of $\mathrm{C}^*(\mathcal{S}_{G,P})$ onto $\mathrm{C}^*(\mathcal{G})$ is not faithful, as it can be seen on very simple examples which don't have lattice properties. For instance for $G = \mathbf{Z}^2$ and $P = \{(t_1 t_2) \in \mathbf{Z}^2 \mid 0 \leq t_2 \leq 2t_1\}$, the identification of Z inside $\{0,1\}^G$ contains elements which are not in Ω, and which are essentially produced by intersections of lines $\{t_2 = a\} \cap \{2t_1 - t_2 = b\}$ with $a, b \in \mathbf{N}$ of different parity.

6.6 Remark Let $\mathcal{S}$ be an $\widetilde{F}$-inverse semigroup. There always exists a "canonical" action of $\mathcal{S}$ which can be considered; namely, $\mathcal{S}$ acts by conjugation on the spectrum of $\mathrm{C}^*(\mathcal{S}^{(o)})$, with $\mathcal{S}^{(o)}$ the subsemigroup of idempotents of $\mathcal{S}$ (see [10], Section 3). Indeed, $\mathcal{S}$ is first seen to act by conjugation on $\mathrm{C}^*(\mathcal{S}^{(o)})$ (this being an action by not necessarily unital $\star$-endomorphisms); passing to the space Z of characters of $\mathrm{C}^*(\mathcal{S}^{(o)})$, we obtain an action $\Phi : \mathcal{S} \to \mathcal{I}_Z$ described as follows:

$$\left\{ \begin{array}{ll} Dom(\Phi(\alpha)) = \{\zeta \in Z \mid \zeta(\alpha^*\alpha) = 1\}, & \alpha \in \mathcal{S} \\ Ran(\Phi(\alpha)) = \{\zeta \in Z \mid \zeta(\alpha\alpha^*) = 1\}, & \alpha \in \mathcal{S} \\ (\Phi(\alpha))(\zeta)(X) \ = \ \zeta(\alpha^* X \alpha) & \alpha \in \mathcal{S}, \zeta \in Dom(\Phi(\alpha)), X \in \mathrm{C}^*(\mathcal{S}^{(o)}). \end{array} \right. \tag{6.8}$$

It is clear that $Dom(\Phi(\alpha))$ is an open and compact subset of Z, for every $\alpha \in \mathcal{S}$, hence the considerations made in this section can be applied. Moreover, the map $\psi^{(0)}$ discussed

in Corollary 6.4 clearly is, in this case, the identity map of Z. Hence if $\mathcal{S}$ is countable, Corollary 6.4 gives that the associated groupoid $\mathcal{G}$ has $\mathrm{C}^*(\mathcal{G}) \simeq \mathrm{C}^*(\mathcal{S})$.

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